GOOGLE CODE-IN 2012	Google Code-In 2012–BRL-CAD: Computing the Volume of a hyp Primative		
Document No.	N/A	Author	Matt S.
Revision	0.0	Date	Dec. 2012

Rev.	**Author**	**Date**	**Summary of Changes**
0.0	Matt S.	07/12/2012	Document created.

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1 A Convoluted Strategy

Assume we have an elliptical hyperboloid, Ω, with it's vertex at some point in $\mathbb{R}^3$ and axis aligned with some vector n. Via a series of three transformations–one translations and two rotations–we can relocate Ω such that the vertex is at $(0, 0, 0)$, with the axis of the cylinder coincident to the x^3 axis, and the major and minor semi-axis of the elliptical cross section alligned with the x^1 and x^2 axis. The result is then something like the shape shown in Figure 1. Consider now an

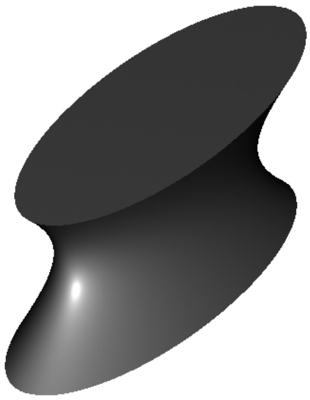

Figure 1: Our Sample Elliptical Hyperboloid, Ω

elliptical cylinder Γ also centered at $(0, 0, 0)$, with a cross section that matches the top of Ω. We can then consider the shape $\Phi = \Gamma \setminus \Omega$, and to make life a little simpler we will only consider the the octant $\left\{x^1, x^2, x^3\right\} \geqslant 0$ and take advantage of the symmetry. We then have the shape illustrated in Figure 2. Consider now the hyperbolic surface of Φ. This surface can be descibed

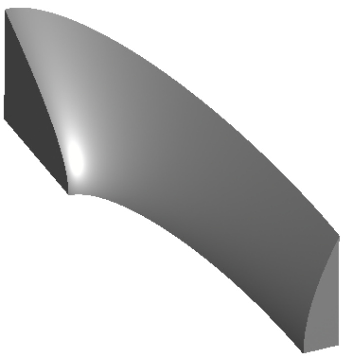

Figure 2: $\Phi = \Gamma \setminus \Omega$, $x \geqslant 0$

via the parametric functions

$$x^i = f^i\left(u^1, u^2\right)$$

where

$$\begin{aligned} f^1\left(u^1, u^2\right) =& a\cosh u^2 \cos u^1 \\ f^2\left(u^1, u^2\right) =& b\cosh u^2 \sin u^1 \\ f^3\left(u^1, u^2\right) =& c\sinh u^2 \end{aligned}$$

Here, a, b, c correspond to the elliptical hyperboloid itself via the fact that Ω is defined via

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

and $u^1 \in \left[0, \frac{\pi}{2}\right]$, $u^2 \in [0, \varphi]$. Note that we may deterimne φ from the height of Ω,

$$h = c\sinh\varphi$$

With this description in place, it's now a fairly straightforward exercise to determine the volume of Φ, which then allows to then determine the volume of Ω, since

$$\Omega = \Gamma \setminus \Phi$$

To actually deterimne the volume of Φ, we first determine the metric tensor g_{ij}. Let

$$\begin{aligned} a^i_j =& \frac{\partial f^i}{\partial u^j} \\ =& \begin{bmatrix} \frac{\partial f^1}{\partial u^1} & \frac{\partial f^2}{\partial u^1} & \frac{\partial f^3}{\partial u^1} \\ \frac{\partial f^1}{\partial u^2} & \frac{\partial f^2}{\partial u^2} & \frac{\partial f^3}{\partial u^2} \end{bmatrix} \\ =& \begin{bmatrix} -a\cosh u^2 \sin u^1 & b\cosh u^2 \cos u^1 & 0 \\ a\sinh u^2 \cos u^1 & b\sinh u^2 \sin u^1 & c\cosh u^2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \Rightarrow g_{ij} =& a^k_i a^k_j \qquad (k \text{ summed}) \\ =& \begin{bmatrix} a^1_1 a^1_1 + a^2_1 a^2_1 + a^3_1 a^3_1 & a^1_1 a^1_2 + a^2_1 a^2_2 + a^3_1 a^3_2 \\ a^1_2 a^1_1 + a^2_2 a^2_1 + a^3_2 a^3_1 & a^1_2 a^1_2 + a^2_2 a^2_2 + a^3_2 a^3_2 \end{bmatrix} \end{aligned}$$

so that we may then compute the volume via

$$V_{compliment} = \int_B \sqrt{\det g_{ij}}\, du^1 du^2$$

where

$$B = [0, \pi/2] \times [0, \varphi]$$

and

$$du^i = \frac{\partial u^i}{\partial x^j} dx^j$$

Of course, this integral will not be terribly simple to compute, but it is very much do-able. Once that's done, we can get the volume of the hyperboloid by simple additon. That is, the volume of the elliptical cylinder is

$$\begin{aligned} V_{cyl} =& \pi abh \\ \Rightarrow V_{hyperboloid} =& V_{cyl} - 8V_{compliment} \end{aligned}$$